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Robustness of controllability under some unbounded perturbations

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Abstract

In this work, we prove that the exact controllability of linear autonomous systems are conserved with "small" Desch–Schappacher perturbations arising, e.g., from the perturbations of dynamic operator's domain. Our results are illustrated by an application to controlled systems with dynamic and boundary perturbations.

Keywords: Exact controllability; Semigroups; Unbounded perturbations; Extrapolation theory; Favard class; Banach space

1. Introduction

In this work we are interested in the following linear control system:

$$(\mathrm{LCS})_P \qquad \begin{cases} \dot{x}(t) = Ax(t) + Px(t) + B(t)u(t), \quad t \geqslant 0, \\ x(0) = x_0, \end{cases}$$

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where the state $x(\cdot)$ takes values in a Banach space X (the state space), the input function $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+, U)$, $p \in [1, \infty)$, and U is a Banach space (the input or control space). The operator $(A, D(A))$ generates a C_0-semigroup on X and the non-autonomous control operator $B(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{L}_s(U, X))$, i.e., $B(t) \in \mathcal{L}(U, X)$ (linear bounded from U to X) for a.e. $t \geqslant 0$ and for each $u \in U$, $B(\cdot)u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, X)$. Here, we consider unbounded perturbations P, in the sense that they take values in a large space than the state space X. Namely, we are interested in the so-called Desch–Schappacher perturbations P and we ask: is the exact controllability of the system $(\mathrm{LCS})_0$ conserved under this kind of perturbations?

In the finite dimensional spaces, Lee and Markus [10, Theorem 2.3.11] has proved that if $(\mathrm{LCS})_0$ (with $B(t) = B$) is exactly controllable then there exists an $\epsilon > 0$ such that for all $\|\tilde{A}\| < \epsilon$ and $\|\tilde{B}\| < \epsilon$, where $\tilde{A}$ and $\tilde{B}$ are matrices of appropriate dimensions, the linear system

$$\dot{x}(t) = (A + \tilde{A})x(t) + (B + \tilde{B})u(t), \quad t \geqslant 0,$$

remains exactly controllable. This means that the controllability of $(\mathrm{LCS})_0$ is not affected by "small" perturbations. In the infinite dimensional spaces, recently, Leiva [11] has considered some class of unbounded perturbations $P : D(A) \to X$ which is not too "irregular" with respect to A. He proved that if $(\mathrm{LCS})_{P_0}$ is exactly controllable then it is so as well for all systems $(\mathrm{LCS})_P$ with P near to P_0 (w.r.t. some metric).

In this work we follow the procedure established by Leiva to prove that the controllability of $(\mathrm{LCS})_0$ is not affected by the class of Desch–Schappacher perturbations P in some neighborhood of zero. More precisely, the perturbation operator P is an operator from the state space X to its extrapolation space X_{-1} (see Section 2). This kind of perturbations arise, e.g., when we have a multiplication perturbation of the dynamic operator A of the form $A(\mathrm{Id} + Q)$ with domain $(\mathrm{Id} + Q)^{-1}(D(A))$, where $Q \in \mathcal{L}(X)$ (see the application in Section 4). We say that the domain is perturbed by Q. In this case, the operator $P = A_{-1}Q$, where A_{-1} is the extrapolated operator of A (see Section 2). In general these perturbations appear in many important applications, e.g., population dynamic models, delay differential equations and generally in boundary Cauchy problems. We refer to [7, Section III.3] and [5] for abstract results concerning this kind of perturbations and to [1,2,12,14,15] for recent applications.

The paper is organized as follows. In Section 2, we recall some notions of extrapolation theory and collect some necessary perturbation results. Section 3 is devoted to study the exact controllability of the system $(\mathrm{LCS})_P$ and we show that this property is conserved when the operator P is close enough to 0 with respect to some metric. In Section 4, we illustrate our framework by an application to the following system with dynamic and boundary perturbations:

$$(\mathrm{CLS})_{D,L} \quad \begin{cases} \dot{x}(t) = A_m x(t) + Dx(t) + B(t)u(t) & \text{for } t \geqslant 0, \\ \Gamma x(t) = Lx(t) & \text{for } t \geqslant 0, \\ x(0) = x_0. \end{cases}$$

We assume that the restriction $A := A_m|_{\ker(\Gamma)}$ generates a C_0-semigroup on a Banach space X. The precise definitions of operators and spaces are given in Section 4. Under some conditions, the operator $C := A_m|_{\ker(\Gamma - L)}$ generates also a C_0-semigroup on X. In concrete applications, the form of semigroup generated by C can not be explicit, which complicates the study of exact controllability of its corresponding controlled system. Hence, it is natural to study the exact controllability of the homogeneous system $(\mathrm{CLS})_{0,0}$ governed by A, and then verify how it is conserved for the perturbed system $(\mathrm{CLS})_{D,L}$. We show that the problem fits in our setting, and we prove that under small perturbations L and D the controllability of the system $(\mathrm{CLS})_{0,0}$ is preserved.

2. Class of Desch–Schappacher perturbations

In this section we give some background on the class of Desch–Schappacher operators. This class was first introduced by Desch and Schappacher (see [5, Section 3.d]) but here we shall consider the large class introduced recently by Engel and Nagel [7, Section III.3] which slightly generalizes the first one. For more details and properties we refer to the mentioned references.

Hereafter, $(A, D(A))$ is the generator of a C_0-semigroup $T(\cdot) := (T(t))_{t\geqslant 0}$ on the Banach space X. We introduce the new norm

$$\|x\|_{-1} := \left\|(\lambda I - A)^{-1}x\right\|$$

for some $\lambda \in \rho(A)$ (the resolvent set of A). The completion of X with respect to this norm is called the extrapolation space associated to X and $T(\cdot)$ (or A). We denote this space by X_{-1} (or X_{-1}^A). Remark that, the norms $\|\cdot\|_{-1}$ are equivalent on X w.r.t. $\lambda \in \rho(A)$, hence the space X_{-1} is independent of the choice of λ. We can see that the space X is the extrapolation space of X_1 (the domain $D(A)$ endowed with the graph norm) w.r.t. the part of A on X_1. Since $T(t)$ commutes with the operator resolvent $R(\lambda, A) := (\lambda I - A)^{-1}$, the extension of $T(t)$ on X_{-1} exists and defines a C_0-semigroup $(T_{-1}(t))_{t\geqslant 0}$ which is generated by A_{-1} with $D(A_{-1}) = X$. For more details and references on extrapolation theory we refer, e.g., to [7, Chapter II.5].

Now, let $T > 0$ and consider the Banach space

$$\mathcal{X}_T := C\big([0,T], \mathcal{L}_s(X)\big) := \big\{F : [0,T] \to \mathcal{L}(X) \text{ strongly continuous}\big\}$$

equipped with the norm

$$\|F\|_\infty := \sup\nolimits_{r\in[0,T]} \left\|F(r)\right\|_{\mathcal{L}(X)}.$$

For $T = \infty$, we consider also the space $\mathcal{X}_\infty := C(\mathbb{R}_+, \mathcal{L}_s(X))$ ($\mathbb{R}_+ := [0,\infty)$). For a given operator $P \in \mathcal{L}(X, X_{-1})$ ($X_{-1} := X_{-1}^A$) we define the *abstract Volterra operator* $V^P : \mathcal{X}_\infty \ni F \mapsto V^P F$, where

$$\big((V^P F)(t)\big)(x) := (V^P F)(t,x) := \int_0^t T_{-1}(t-r)PF(r)x\,dr \quad \text{for } x \in X,\ t \geqslant 0.$$

We remark that the operator V^P is causal, i.e., $V^P(F|_{[0,T]}) = (V^P F)|_{[0,T]}$, $T > 0$. We denote by V_T^P its restriction on $\mathcal{X}_T$ which belongs to $\mathcal{L}(C([0,T], \mathcal{L}_s(X, X_{-1}))$. Since the operator $(V^P F)(t)$ may take values in the extrapolation space X_{-1}, we introduce the set

$$\mathcal{S}_T(A) := \big\{P \in \mathcal{L}(X, X_{-1}):\ V_T^P \in \mathcal{L}(\mathcal{X}_T) \text{ and } \left\|V_T^P\right\|_{\mathcal{L}(\mathcal{X}_T)} < 1\big\}.$$

Remark also that one can extend $F \in \mathcal{X}_T$ to obtain an element of $\mathcal{X}_\infty$ by setting $F(t) = F(T)$ for all $t \geqslant T$. Thus, if $V_T^P \in \mathcal{L}(\mathcal{X}_T)$ then $V_t^P \in \mathcal{L}(\mathcal{X}_t)$ and $\|V_t^P\|_{\mathcal{L}(\mathcal{X}_t)} \leqslant \|V_T^P\|_{\mathcal{L}(\mathcal{X}_T)}$ for all $t \leqslant T$. This implies that the sets $\mathcal{S}_T(A)$, $T > 0$, are decreasing by inclusion.

The set $\mathcal{S}_T(A)$ is called the class of *Desch–Schappacher perturbations*. The following result shows that any additive perturbation of the generator A by Desch–Schappacher perturbations still generates a C_0-semigroup. The proof is given in [7, Theorem III.3.1 and Corollary III.3.2].

Theorem 1. *Let $(A, D(A))$ generates a C_0-semigroup $T(\cdot) := (T(t))_{t\geqslant 0}$ on the Banach space X. If $P \in \mathcal{S}_T(A)$ then the operator*

$$A^P x := A_{-1}x + Px \quad \text{with } D(A^P) := \{x \in X\colon\ A_{-1}x + Px \in X\} \tag{1}$$

generates a C_0-semigroup $T^P(\cdot) := (T^P(t))_{t\geqslant 0}$. The semigroup $T^P(\cdot)$ is given by the following variation of constants formula:

$$T^P(t)x = T(t)x + \int_0^t T_{-1}(t-r)PT^P(r)x\,dr \quad \text{for } t \geqslant 0 \text{ and } x \in X, \tag{2}$$

and by the Dyson–Phillips series

$$T^P(t)x := \sum_{n=0}^{\infty} T_n^P(t)x \quad \text{for } t \geqslant 0 \text{ and } x \in X, \tag{3}$$

where $T_0^P(t) := T(t)$ and

$$T_n^P(t)x := \int_0^t T_{-1}(t-r)PT_{n-1}^P(r)x\,dr \quad \text{for } n \in \mathbb{N}^* := \{1, 2, \ldots\}.$$

Here, the series (3) *converges in $\mathcal{L}(X)$ uniformly on compact intervals of $\mathbb{R}_+$.*

In [7, Section III.3], it was given simple and practical conditions on the operator P to belong to $\mathcal{S}_T(A)$ and hence the operator $(A^P, D(A^P))$ is a generator on X.

Proposition 2. *Let $(A, D(A))$ generates a C_0-semigroup $T(\cdot)$ on the Banach space X and let $P \in \mathcal{L}(X, X_{-1})$. Let one of the following conditions* (1) *or* (2) *be satisfied.*

(1) *There exists $T > 0$ and $\alpha \in [0, 1)$ such that*
   (i) *for all $f \in C([0, T], X)$, with $\|f\|_\infty := \sup_{r\in[0,T]}\|f(r)\|$,*

$$\int_0^T T_{-1}(T-r)Pf(r)\,dr \in X \quad \text{and} \tag{4}$$

   (ii) $$\left\| \int_0^T T_{-1}(T-r)Pf(r)\,dr \right\| \leqslant \alpha\|f\|_\infty.$$

(2) *There exists* $T > 0$ *and* $p \in [1, \infty)$ *such that* (4) *holds for all* $f \in L^p(0, T; X)$.

Then, the operator $P \in \mathcal{S}_T(A)$.

Remark 3. (i) If the operator $P \in \mathcal{L}(X, X_{-1})$ satisfies (4), for some $T > 0$, then $PQ \in \mathcal{S}_T(A)$ for all $Q \in \mathcal{L}(X)$. In general, if $P \in \mathcal{S}_T(A)$, for some $T > 0$, and $Q \in \mathcal{L}(X)$ with $\|Q\| \leqslant 1$, then $PQ \in \mathcal{S}_T(A)$ since

$$\left\|V_T^{PQ}\right\|_{\mathcal{L}(\mathcal{X}_T)} \leqslant \|Q\|_{\mathcal{L}(X)}\left\|V_T^P\right\|_{\mathcal{L}(\mathcal{X}_T)}. \tag{5}$$

(ii) Let $P \in \mathcal{L}(X, X_{-1})$ with range in the Favard class $F_{A_{-1}}$ (in particular, $P \in \mathcal{L}(X)$). Then, P verifies the condition (4) of the above proposition with $p = 1$ and hence $P \in \mathcal{S}_T(A)$ for some $T > 0$, see [7, Corollary III.3.6].

We recall that the Favard class associated to a generator A (or $T(\cdot)$) is the Banach space

$$F_A := \left\{x \in X\colon \sup_{t>0} \frac{1}{t}\left\|e^{-\omega t}T(t)x - x\right\| < \infty\right\}$$

endowed with the norm

$$\|x\|_{F_A} := \sup_{t>0} \frac{1}{t}\left\|e^{-\omega t}T(t)x - x\right\|.$$

Here $\omega > \omega_0(T(\cdot))$ fixed ($\omega_0(T(\cdot))$ is the growth bound of the semigroup $T(\cdot)$). We note that F_A is independent of the choice of ω, contains the domain of A and isomorph to $F_{A_{-1}}$ since $(\lambda - A_{-1})F_A = F_{A_{-1}}$, $\lambda \in \rho(A)$. In the case when X is a reflexive Banach space, the Favard class associated to $T(\cdot)$ is exactly the domain of its generator (see, e.g., [7, Section II.5.b] or [3, Chapter 3] for more properties).

Here, we give a useful exponential estimate of the abstract Volterra operator.

Proposition 4. *Let* $M \geqslant 1$ *and* $\omega \in \mathbb{R}$ *such that* $\|T(t)\| \leqslant Me^{\omega t}$ *for all* $t \geqslant 0$. *Let* $P \in \mathcal{S}_{T_0}(A)$ *for some* $T_0 > 0$. *Then, for all* $t \geqslant 0$, *and* $F \in \mathcal{X}_\infty$ *we have*

(i) $(V^P F)(t) \in \mathcal{L}(X)$ *and*

(ii) $$\left\|(V^P F)(t)\right\|_{\mathcal{L}(X)} \leqslant N_\omega(t)\left\|V_{T_0}^P\right\|_{\mathcal{L}(\mathcal{X}_{T_0})}\|F\|_{\mathcal{X}_t}, \tag{6}$$

where

$$N_\omega(t) := \begin{cases} n(\omega)e^{\omega t} & \textit{if } \omega > 0, \\ M\left(\frac{t}{T_0} + 1\right) & \textit{if } \omega = 0, \\ n(\omega) & \textit{if } \omega < 0, \end{cases}$$

and

$$n(\omega) := M\frac{e^{|\omega|T_0}}{|e^{\omega} - 1|}.$$

Proof. Let $F \in \mathcal{X}_\infty$ and $t \geqslant \tau \geqslant 0$. Then, the integration by part of $(V^P F)(t)$ gives

$$(V^P F)(t) = T_{-1}(t-\tau)(V^P F)(\tau) + (V^P F_\tau)(t-\tau), \tag{7}$$

where $F_\tau := F(\cdot + \tau)$. Take instead of (t, τ) the pair (nt, t) with $n \in \mathbb{N}^*$. Then, we have

$$(V^P F)(nt) = T_{-1}\big((n-1)t\big)(V^P F)(t) + (V^P F_t)\big((n-1)t\big). \tag{8}$$

By applying the same procedure to the second term of the right-hand side of (8) with the pair $((n-1)t, t)$ and so on we obtain finally, by induction, that

$$\begin{aligned}(V^P F)(nt) = {} & T_{-1}\big((n-1)t\big)(V^P F)(t) \\ & + T_{-1}\big((n-2)t\big)(V^P F_t)(t) + \cdots + (V^P F_{(n-1)t})(t)\end{aligned} \tag{9}$$

for all $n \in \mathbb{N}^*$ and $t \geqslant 0$.

Now, let $t \in \,](n_0-1)T_0, n_0 T_0]$, where $n_0 := 2, 3, \ldots$ (the case $t \in [0, T_0]$ is trivial, since $N_w(t) \geqslant 1$), and set $\tau := t - (n_0-1)T_0$. Applying successively (7) and (9) we deduce

$$\begin{aligned}(V^P F)(t) &= T_{-1}\big((n_0-1)T_0\big)(V^P F)(\tau) + (V^P F_\tau)\big((n_0-1)T_0\big) \\ &= T\big((n_0-1)T_0\big)\big(V^P_{T_0} F\big)(\tau) \\ &\quad + T\big((n_0-2)T_0\big)\big(V^P_{T_0} F_\tau\big)(T_0) + \cdots + \big(V^P_{T_0} F_{t-T_0}\big)(T_0)\end{aligned} \tag{10}$$

and this shows the first assertion. By taking the $\mathcal{L}(X)$-norm in (10) we obtain

$$\begin{aligned}\big\|(V^P F)(t)\big\| &\leqslant \big\|T\big((n_0-1)T_0\big)\big\|\,\big\|\big(V^P_{T_0} F\big)(\tau)\big\| \\ &\quad + \big\|T\big((n_0-2)T_0\big)\big\|\,\big\|\big(V^P_{T_0} F_\tau\big)(T_0)\big\| + \cdots + \big\|\big(V^P_{T_0} F_{t-T_0}\big)(T_0)\big\| \\ &\leqslant \big\|T\big((n_0-1)T_0\big)\big\|\,\big\|V^P_{T_0}\big\|\,\|F\|_{\mathcal{X}_{T_0}} \\ &\quad + \sum_{k=0}^{n_0-2} \big\|T(kT_0)\big\|\,\big\|V^P_{T_0}\big\|\,\|F_{t-(k+1)T_0}\|_{\mathcal{X}_{T_0}} \\ &\leqslant \left(\sum_{k=0}^{n_0-1} \big\|T(kT_0)\big\|\right)\big\|V^P_{T_0}\big\|\,\|F\|_{\mathcal{X}_t}.\end{aligned} \tag{11}$$

Thus, if $\omega \neq 0$ we obtain

$$\big\|(V^P F)(t)\big\| \leqslant M\frac{e^{\omega(n_0 T_0)}-1}{e^\omega - 1}\big\|V^P_{T_0}\big\|\,\|F\|_{\mathcal{X}_t} \leqslant N_\omega(t)\big\|V^P_{T_0}\big\|\,\|F\|_{\mathcal{X}_t},$$

and if $\omega = 0$ we obtain, from (11), that

$$\big\|(V^P F)(t)\big\| \leqslant M n_0 \big\|V^P_{T_0}\big\|\,\|F\|_{\mathcal{X}_t} \leqslant M\left(\frac{t}{T_0}+1\right)\big\|V^P_{T_0}\big\|\,\|F\|_{\mathcal{X}_t}. \qquad \square$$

3. The exact controllability

Consider the following linear control system:

$$(\mathrm{LCS})_0 \quad \begin{cases} \dot{x}(t) = Ax(t) + B(t)u(t), & t \geqslant 0, \\ x(0) = x_0, \end{cases}$$

where the state $x(\cdot)$ takes values in a Banach space X, the input function $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+, U)$, $p \in [1, \infty)$, with U is a Banach space. The operator $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\geqslant 0}$ on X and the non-autonomous control operator $B(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{L}_s(U, X))$.

We begin first by recalling the notion of mild solutions. It is known that any (classical) solution of the system $(\mathrm{LCS})_0$ is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-r)B(r)u(r)\,dr \quad \text{for } t \geqslant 0. \tag{12}$$

The function $x(\cdot, x_0, u) \in C([0, \infty), X)$ verifying (12) is called the mild solution of $(\mathrm{LCS})_0$ associated to data $x_0 \in X$ and $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+, U)$, where $p \in [1, \infty)$.

Definition 5. The system $(\mathrm{LCS})_0$ is called exactly p-controllable on $[0, T]$, with $T > 0$, if for all x_0 and x_1 in X there exists $u \in L^p(0, T; U)$ such that the mild solution x of $(\mathrm{LCS})_0$ verifies $x(T, x_0, u) = x_1$.

This means that starting from any initial state one can reach any finale state in the space X at time T by choosing some suitable input for the system $(\mathrm{LCS})_0$. An other reformulation of the exact p-controllability is by means of the so-called *controllability map* $\mathcal{C}_0 \in \mathcal{L}(L^p(0, T; U), X)$ defined by

$$\mathcal{C}_0 u := \int_0^T T(T-r)B(r)u(r)\,dr. \tag{13}$$

Proposition 6. *The system* $(\mathrm{LCS})_0$ *is exactly* p*-controllable on* $[0, T]$ *if and only if the controllability map* $\mathcal{C}_0$ *is surjective. In particular, when the state space* X*, the input space* U *are reflexive Banach spaces, and* $p \in\,]1, \infty)$*, the exact* p*-controllability of* $(\mathrm{LCS})_0$ *is thus equivalent to*

$$\beta \left\| B^*(\cdot)T^*(\cdot)x^* \right\|_{L^q(0,T;U^*)} \geqslant \|x^*\| \quad \text{for } x^* \in X^* \ (X^* \text{ is the topological dual of } X), \tag{14}$$

where $1/p + 1/q = 1$ *and the constant* $\beta > 0$.

Proof. It is straightforward to show the first equivalence. The second can be found, e.g., in [4, Chapter 4] or [7, p. 458]. □

Now, let us consider an operator $P \in \mathcal{S}_T(A)$. The mild solution of the perturbed control system $(\mathrm{LCS})_P$ is the function $x(\cdot) := x(\cdot, x_0, u) \in C([0, \infty), X)$ given by

$$x(t) = T^P(t)x_0 + \int_0^t T^P(t-r)B(r)u(r)\,dr \quad \text{for } t \geqslant 0 \tag{15}$$

which also verifies the integral equation

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-r)Px(r)\,dr + \int_0^t T(t-r)B(r)u(r)\,dr \\ \text{for } t \geqslant 0. \tag{16}$$

In particular, if $P \in \mathcal{L}(X, F_{A_{-1}})$ then the mild solution $x(\cdot)$ of $(\mathrm{LCS})_P$ is the unique solution of the integral equation (16).

We define, for each $T > 0$, the map

$$d_T : \mathcal{S}_T(A) \times \mathcal{S}_T(A) \ni (P, Q) \mapsto \left\| V_T^{P-Q} \right\|_{\mathcal{L}(\mathcal{X}_T)}.$$

Then, d_T defines a metric on the set $\mathcal{S}_T(A)$. One can see that $\mathcal{S}_T(A)$ is not complete with respect to d_T and if the difference $P - Q \in \mathcal{L}(X)$ then there exists a constant $c = c(T)$ such that

$$d_T(P, Q) \leqslant c \, \|P - Q\|_{\mathcal{L}(X)}.$$

The following result gives an important tool by showing that the subset of surjective operators is open in the space of bounded linear operators. The proof can be found in [9, p. 227].

Theorem 7. *Let E and F be Banach spaces and let $\mathcal{C}_0 \in \mathcal{L}(E, F)$ be surjective. Then, there exists $\alpha > 0$ such that for all $\mathcal{C} \in \mathcal{L}(E, F)$ with $\|\mathcal{C} - \mathcal{C}_0\|_{\mathcal{L}(E,F)} < \alpha$ the operator $\mathcal{C}$ is also surjective.*

The constant α is called the surjectivity-ray of $\mathcal{C}_0$. To prove the main result of this section we need the following lemmas. We begin first by giving an explicit exponential estimate of the semigroup $T^P(\cdot)$.

Lemma 8. *Let $P \in \mathcal{S}_{T_0}(A)$ for some $T_0 > 0$ and set $M_{T_0} := \sup_{t \in [0, T_0]} \|T(t)\|$. Then, we have*

$$\left\| T^P(t) \right\| \leqslant M_P e^{\omega_P t} \quad \textit{for all } t \geqslant 0, \tag{17}$$

where

$$M_P := \frac{M_{T_0}}{1 - \|V_{T_0}^P\|_{\mathcal{L}(\mathcal{X}_{T_0})}} \quad \textit{and} \quad \omega_P := \frac{\log(M_P)}{T_0}.$$

Proof. From the Dyson–Phillips series (3), we obtain

$$\left\| T^P(t) \right\| \leqslant M_P \quad \text{for all } t \in [0, T_0].$$

Hence, for $t \in [nT_0, (n+1)T_0)$, where n is a positive integer, we obtain

$$\left\| T^P(t) \right\| \leqslant M_P^{n+1}.$$

Thus, the estimate (17) follows. □

In the following lemma, we give an estimation of the difference of the two semigroups $T(\cdot)$ and $T^P(\cdot)$ which is crucial to get our aim.

Lemma 9. *Let $P \in \mathcal{S}_{T_0}(A)$ for some $T_0 > 0$. Then, we have*

$$\left\|T^P(t) - T(t)\right\| \leqslant M_{T_0} N_\omega(t) \frac{d_{T_0}(P,0)}{1 - d_{T_0}(P,0)} \left(\frac{M_{T_0}}{1 - d_{T_0}(P,0)}\right)^{t/T_0} \tag{18}$$

for all $t \geqslant 0$, where $N_\omega(\cdot)$ is the function given in Proposition 4.

Proof. According to the variation of constant formula (2), we have

$$T^P(t)x - T(t)x = \int_0^t T_{-1}(t-s) P T^P(s) x\, ds = V^P\big(T^P(\cdot)\big)(t,x)$$

for all $x \in X$ and $t \geqslant 0$. Thus, the lemma can be deduced from Lemma 8 and Proposition 4. □

Now, we can state the main result of this section which shows the robustness of exact controllability under small perturbations.

Theorem 10. *Let $P \in \mathcal{S}_{T_0}(A)$ for some $T_0 > 0$ and assume that the unperturbed linear control system* $(\mathrm{LCS})_0$ *is exactly p-controllable on $[0,T]$ for $T > 0$ and $p \in [1,\infty)$. Then, there exists a neighborhood $\mathcal{N}(0)$ of zero in the set $\mathcal{S}_{T_0}(A)$ such that for each $P \in \mathcal{N}(0)$ the system* $(\mathrm{LCS})_P$ *is also exactly p-controllable on* $[0,T]$.

Proof. Let $u \in L^p([0,T],U)$. By Lemma 9, we have

$$\begin{aligned}
\|\mathcal{C}_0 u - \mathcal{C}_P u\| &= \left\| \int_0^T \big(T^P(t-s) - T(t-s)\big) B(s) u(s)\, ds \right\| \\
&\leqslant \int_0^T \left\|T^P(t-s) - T(t-s)\right\| \left\|B(s)u(s)\right\| ds \\
&\leqslant \|B\|_{L^\infty(0,T,\mathcal{L}_s(U,X))} M_{T_0} N_\omega(T) \frac{d_{T_0}(P,0)}{1 - d_{T_0}(P,0)} \\
&\quad \times \left(\frac{M_{T_0}}{1 - d_{T_0}(P,0)}\right)^{T/T_0} \int_0^T \left\|u(s)\right\| ds.
\end{aligned}$$

Therefore, by Hölder's inequality, it follows that

$$\|\mathcal{C}_0 u - \mathcal{C}_P u\| \leqslant \beta f\big(d_{T_0}(P,0)\big) \|u\|_{L^P([0,T],U)}, \tag{19}$$

where

$$\beta := N_\omega(T) T^{(p-1)/p} \|B\|_{L^\infty(0,T;\mathcal{L}_s(U,X))} M_{T_0}$$

and

$$f(r) := \frac{r}{1-r}\left(\frac{M_{T_0}}{1-r}\right)^{T/T_0} \quad \text{for } r \in [0,1).$$

The function f is continuous from $[0,1)$ to $\mathbb{R}_+$ and strictly increasing, hence it is bijective. We denote its inverse by f^{-1}. On the other hand, we have $\mathcal{C}_0$ is surjective by assumption. Let α be the surjectivity-ray associated to $\mathcal{C}_0$. Then, via Theorem 7, $\mathcal{C}_P$ is surjective when $\|\mathcal{C}_0 - \mathcal{C}_P\| < \alpha$. In particular, by (19), this occurs if $d_{T_0}(P,0) < f^{-1}(\alpha/\beta)$. Thus, one can take the neighborhood $\mathcal{N}(0)$ to be the open ball $\mathcal{B}(0, f^{-1}(\alpha/\beta))$ in $\mathcal{S}_{T_0}(A)$ with the center 0 and the ray $f^{-1}(\alpha/\beta)$. □

In Remark 3(i), we have seen that if $P \in \mathcal{L}(X, X_{-1})$ such that $V_{T_0}^P \in \mathcal{L}(\mathcal{X}_{T_0})$, for some $T_0 > 0$, and $Q \in \mathcal{L}(X)$ then, $V_{T_0}^{PQ} \in \mathcal{L}(\mathcal{X}_{T_0})$ and

$$\left\| V_{T_0}^{PQ} \right\|_{\mathcal{L}(\mathcal{X}_{T_0})} \leqslant \|Q\|_{\mathcal{L}(X)} \left\| V_{T_0}^{P} \right\|_{\mathcal{L}(\mathcal{X}_{T_0})}.$$

Thus, one can deduce the following result from Theorem 10.

Corollary 11. *Assume that the system* $(\mathrm{LCS})_0$ *is exactly* p*-controllable on* $[0,T]$ *and let* $P \in \mathcal{S}_{T_0}(A)$ *for some* $T_0 > 0$*. Then, there exists* $q \in\,]0,1]$ *such that the system* $(\mathrm{LCS})_{PQ}$ *is exactly* p*-controllable on* $[0,T]$ *for any operator* $Q \in \mathcal{L}(X)$ *such that* $\|Q\| < q$.

4. An application

An abstract application of our framework is to consider the controlled linear system with boundary condition of the following type:

$$(\mathrm{CLS})_{D,L} \qquad \begin{cases} \dot{x}(t) = A_m x(t) + Dx(t) + B(t)u(t) & \text{for } t \geqslant 0, \\ \Gamma x(t) = Lx(t) & \text{for } t \geqslant 0, \\ x(0) = x_0. \end{cases}$$

Here $(A_m, D(A_m))$ is a densely defined linear operator on a Banach space X, $\Gamma : D(A_m) \to \partial X$, where the boundary space ∂X is a Banach space, $L : \mathcal{L}(X, \partial X)$ is a bounded linear operator and $D \in \mathcal{L}(X)$. We shall make the following assumptions used by Greiner [8] for studying the boundary perturbations of generators.

(A1) There exists a new norm $|\cdot|$ which makes the domain $D(A_m)$ complete and then denoted by X_m. The space X_m is continuously embedded in X and $A_m \in \mathcal{L}(X_m, X)$.
(A2) $\Gamma \in \mathcal{L}(X_m, \partial X)$ is surjective.
(A3) The restriction $A := A_m|_{\ker(\Gamma)}$ generates a C_0-semigroup $T(\cdot)$ on X.

One can see that under (A1)–(A2) and for some $\lambda \in \rho(A)$ the maximal domain X_m can be decomposed as follows:

$$X_m = \operatorname{Ker} \Gamma \oplus \ker(\lambda - A_m). \tag{20}$$

Thus, the restriction $\varGamma : \mathrm{Ker}(\lambda - A_m) \to \partial X$ is then a bijection with the inverse is the so-called *Dirichlet operator* $\varGamma_\lambda \in \mathcal{L}(\partial X, X)$ and $\varGamma_\lambda \varGamma$ is a projection onto $\ker(\lambda - A_m)$.

In the following, we assume (A1)–(A3) hold and set $P := (\lambda - A_{-1})\varGamma_\lambda L$ for some fixed $\lambda \in \rho(A)$. Remark that P is independent of the choice of λ since we have the property $\varGamma_\lambda = (I - (\lambda - \mu)R(\lambda, A))\varGamma_\mu$ for all $\lambda, \mu \in \rho(A)$ (see [8, Lemma 1.3]). The following lemma gives a relation between the maximal operator A_m and the extrapolated generator A_{-1}.

Lemma 12. *Assume that* (A1)–(A3) *are satisfied. Let $\lambda \in \rho(A)$ and let $x \in X$ such that $x - \varGamma_\lambda Lx \in D(A)$. Then we have $x \in X_m$ and*

$$A_m x = A_{-1}x + Px.$$

Proof. Let x be as in lemma. Since $\mathrm{Range}(\varGamma_\lambda) \subset \mathrm{Ker}(\lambda - A_m)$, we have $x \in X_m$ and

$$\begin{aligned}(\lambda - A_m)x &= (\lambda - A_m)(x - \varGamma_\lambda Lx) = (\lambda - A)(x - \varGamma_\lambda Lx)\\ &= (\lambda - A_{-1})(x - \varGamma_\lambda Lx).\end{aligned}$$

Thus, the result follows. □

We introduce the operator

$$C := A_m \quad \text{with } D(C) := \{x \in X_m,\ \varGamma x = Lx\}.$$

The following result shows that C and the operator A^P, as defined in (1), are the same.

Proposition 13. *Assume that* (A1)–(A3) *are satisfied. Then, we have $C = A^P$.*

Proof. Let $x \in D(A^P)$. Then, one can write

$$A_{-1}x + Px = A_{-1}(x - \varGamma_\lambda Lx) + \lambda \varGamma_\lambda Lx$$

which belongs to X. By using the fact that

$$A_{-1}z \in X \quad \text{if and only if} \quad z \in D(A), \tag{21}$$

it then follows that $x - \varGamma_\lambda Lx \in D(A)$ and hence $\varGamma x = Lx$. Thus, by Lemma 12, we obtain $x \in D(C)$ and $Cx = A^P x$. The converse can be deduced by applying again Lemma 12. □

From the above proposition and Theorem 1, the operator C generates a C_0-semigroup if $P \in \mathcal{S}_{T_0}(A)$ for some $T_0 > 0$. This occurs, in particular, if X_m is included in the Favard class F_A, see Remark 3(ii). In [6, p. 540], it was showed that this last condition holds if and only if there exists positive constants γ, λ_0 such that

$$\|\varGamma_\lambda\| \leqslant \frac{\gamma}{\lambda - \lambda_0} \tag{22}$$

for all $\lambda \in \rho(A)$ such that $\lambda > \lambda_0$.

Now, we turn back to the problem of controllability. We begin by the following result about the exact controllability of the system $(\mathrm{CLS})_{0,L}$.

Proposition 14. *Under the assumptions* (A1)–(A3)*, we assume also that* (22) *holds. If the homogeneous system* $(\mathrm{CLS})_{0,0}$ *is exactly* p*-controllable on* $[0,T]$ *for* $T>0$ *and* $p\in[1,\infty)$*, then there exists* $l>0$ *such that the system* $(\mathrm{CLS})_{0,L}$ *is also exactly* p*-controllable on* $[0,T]$ *for any boundary perturbations* $\|L\|_{\mathcal{L}(X,\partial X)}<l$.

Proof. From (22), the operator $P\in\mathcal{S}_{T_0}(A)$, for some $T_0>0$. Let $F\in\mathcal{X}_{T_0}$, $t\in[0,T_0]$ and $x\in X$. Then, we have

$$\big(V_{T_0}^P F\big)(t,x)=\int_0^t T_{-1}(t-r)PF(r)x\,dr=(\lambda-A_{-1})\int_0^t T(t-r)\Gamma_\lambda LF(r)x\,dr$$
$$=(\lambda-A)\int_0^t T(t-r)\Gamma_\lambda LF(r)x\,dr.$$

This last equality follows from (21). Now, by the open mapping theorem, we have $\Gamma_\lambda\in\mathcal{L}(\partial X,F_A)$. Applying [5, Theorem 9] (see also [13, Proposition 3.3]), we then obtain

$$\begin{aligned}\big\|\big(V_{T_0}^P F\big)(t,x)\big\| &\leqslant \nu\|\lambda-A\|_{\mathcal{L}(X_1,X)}\exp(\omega t)\big\|\Gamma_\lambda LF(\cdot)x\big\|_{L^1(0,t;F_A)}\\ &\leqslant N(\lambda,T_0)\|\Gamma_\lambda\|_{\mathcal{L}(\partial X,F_A)}\|L\|_{\mathcal{L}(X,\partial X)}\|F\|_{\mathcal{X}_{T_0}}\|x\|,\end{aligned}$$

where X_1 is the domain $D(A)$ endowed with the graph norm and

$$N(\lambda,T_0):=\nu T_0\|\lambda-A\|_{\mathcal{L}(X_1,X)}\exp(\omega T_0)\quad\text{with } \nu,\ \omega>0.$$

This means

$$\big\|V_{T_0}^P\big\|_{\mathcal{L}(\mathcal{X}_{T_0})}\leqslant N(\lambda,T_0)\|\Gamma_\lambda\|_{\mathcal{L}(\partial X,F_A)}\|L\|_{\mathcal{L}(X,\partial X)}.$$

Thus, the proposition can be deduced from Theorem 10. □

Finally, we consider the system $(\mathrm{CLS})_{D,L}$ where the dynamic operator is

$$A_{D,L}:=C+D\quad\text{with } D(A_{D,L}):=D(C).$$

This operator generates a C_0-semigroup on X provided that C generates a C_0-semigroup, in particular if (22) holds. The following result shows that the system $(\mathrm{CLS})_{D,L}$ remains exactly controllable for small perturbations D and L.

Theorem 15. *Under the assumptions of Proposition* 14*, if the homogeneous system* $(\mathrm{CLS})_{0,0}$ *is exactly* p*-controllable on* $[0,T]$ *for* $T>0$ *and* $p\in[1,\infty)$*, then there exists* $d>0$ *such that the system* $(\mathrm{CLS})_{D,L}$ *is also exactly* p*-controllable on* $[0,T]$ *for any perturbations* $\|D\|_{\mathcal{L}(X)}<d$ *and* $\|L\|_{\mathcal{L}(X,\partial X)}<l$*, where* $l>0$ *is given in Proposition* 14.

Proof. This follows from Proposition 14 and Corollary 11 by taking in this latter, C, Id and D instead of A, P and Q, respectively. □

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